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Emergence of order in dynamical phases in coupled fractional gauss map

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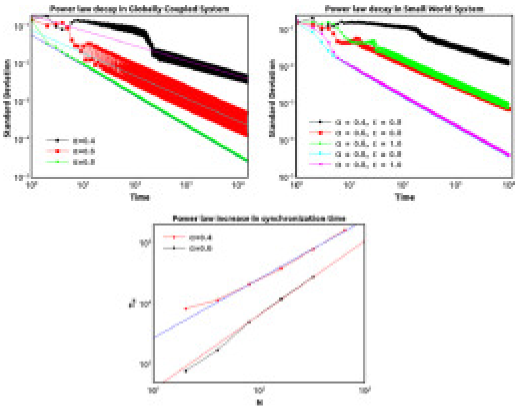
Highlights

- We define a spatiotemporal system with long-term memory in time. We extend the definition of fractional order maps to coupled map lattice.
- We observe synchronized periodic states with period 3 or 6 even in one dimension.
- These states are possible in the thermodynamic limit.
- In the synchronized state, the standard deviation decays as a power-law and power is related to the fractional-order of maps.

Abstract

Dynamical behaviour of discrete dynamical systems has been investigated extensively in the past few decades. However, in several applications, long term memory plays an important role in the evolution of dynamical variables. The definition of discrete maps has recently been extended to fractional maps to model such situations. We extend this definition to a spatiotemporal system. We define a coupled map lattice on different topologies, namely, one-dimensional coupled map lattice, globally coupled system and small-world network. The spatiotemporal patterns in the fractional system are more ordered. In particular, synchronization is observed over a large parameter region. For integer order coupled map lattice in one dimension, synchronized periodic states with a period greater than one are not obtained. However, we observe synchronized periodic states with period-3 or period-6 in one dimensional coupled fractional maps even for a large lattice. With nonlocal coupling, the synchronization is reached over a larger parameter regime. In all these cases, the standard deviation decays as power-law in time with the power same as fractional-order. The physical significance of such studies is also discussed.

Graphical abstract



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Introduction

Discrete fractional calculus has a long history starting with Leibniz (1695) and is touched upon by almost every eminent mathematician. The recent revival of interest in fractional calculus is due to the successful application of fractional calculus in different fields such as quantum mechanics, electromagnetics, bioengineering, signal processing and many more [1], [2], [3], [4], [5], [6]. Both low-dimensional, as well as high dimensional spatiotemporal systems, are studied in this context. Fractional partial differential equations are also proposed [7], [8], [9]. In this paper, we propose a system of coupled fractional Gauss maps and study the phase transitions in it. In coupled map lattice, we study finite-dimensional difference equations coupled by Laplace operator. We generalize the difference equations to fractional-order.

Several systems in nature have inherent long term memory in it [10], [11]. Rheological systems, geophysical systems or fracture dynamics have been modelled by integrodifferential equations for a long time [12], [13], [14], [15], [16], [17]. Fractional differential equations offer an alternative modeling scheme [18], [19]. By now, we know several cases where fractional differential equations have been successful in providing a qualitative and quantitative understanding of the system [20], [21].

Fractional calculus has been successfully applied to spatiotemporal systems as well. Anomalous random walks have been observed experimentally since 1921 and several theories have been proposed to understand those. One of the recent models uses fractional diffusion equations to model it [22], [23], [24]. The fractional diffusion equations have also been useful in modelling the behaviour of random walkers in an expanding medium [25].

In compressional and shear waves in sediments, a near-linear variation of attenuation with frequency is observed over some range. The standard Biot theory predicts attenuation that increases and frequency squared or either levelling off or increasing as square root at high frequencies. An alternative theory explaining thus attenuation was proposed by Holm and Pandey. It has time-domain memory operator equivalent to a fractional derivative operator. Holm and Pandey [26]. Contribution of ions to electrical impedance in an electrolytic cell is related to the anomalous diffusion process and memory effects. Evengelista *et al* proposed analytical solutions of fractional diffusion equations to explain it [27]. Decomposition of supersaturated solid solution can lead to the formation of clusters and precipitates of atoms and defects. They can change many material properties in a significant manner. Their kinetics is described by the diffusion-limited process. Sibatov and Svetukhin developed generalized equations of subdiffusion-limited growth using fractional derivatives to model this situation [28]. Another example where fractional calculus has helped to model physical phenomena is heat conduction. Classical Fourier law and heat conduction equation are inadequate for describing heat conduction in several materials. Time-nonlocal generalization using fractional calculus was proposed by Povstenko in this context [29]. Fractional calculus has also been used for spatiotemporal systems such as reaction-diffusion systems [30]. Thus fractional equations have ben useful in modelling several physical systems including spatiotemporal systems.

We can construct a spatiotemporal system which comprises of units which follow fractional dynamics and such system can shed light on dynamics in several naturally occurring spatiotemporal systems.

A system consisting of two coupled fractional Henon maps was proposed in [31] and the synchronization was studied. Though coupled fractional logistic map has been studied in [32], the formulation is the same as usual coupled map lattice. The only difference is that a variant of the logistic map is used as an on-site map (see eq. 5 of [32]). Thus the entire theory of coupled map lattice can be used in this case. Our formulation has a long-term memory which is absent in standard definition of coupled map lattice. We observe that this helps in establishing long-range order even in low dimensional systems. Our definition reduces to a single fractional map in the absence of coupling. We study this formulation for a few topologies such as one-dimensional lattice, global coupling and small-world lattice. The prescription is generic enough to be extended to any underlying topology.

While fractional differential equations have been extremely successful in studies of several physical systems, their numerical simulation is tedious and time-consuming. It is easier to simulate fractional difference equations. We may be able to spot the essential characteristics of the system using such modelling. We expect a generic nature of the dynamical evolution of systems with long-term memory that can be learned using these discrete-time models. There have been very few investigations in such systems or their spatially extended counterpart. We study such systems with local as well as nonlocal coupling. Such modelling may capture some phenomena in continuous time spatiotemporal systems with fractional-order.

From the engineering viewpoint, we note that dynamical systems with long-term memory have been studied in the context of control of chaos. Socolor and co-workers [33] studied systems with extended time-delayed feedback. In this system, the evolution has a component from the feedback signal given by $(1-R)\sum_{j=1}^{t} R^{k-1} x\,(t-k)$. Such feedback has been found extremely useful in controlling certain models as well as experimental systems. In this case, the simulation can be simplified by introducing another variable and converting it to an effectively two-dimensional system since the kernel is exponential. In our case, the kernel is a power-law and such simplification is not available. However, there have been advances in simulating electrical circuits of non-integer order and it is possible that such kernels can be experimentally realized [34].

We follow the definition in [35]. Deshpande and Daftardar-Gejji define a fractional difference operator and define the evolution as,
$x\,(t)=x\,(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{t}\frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)}\left[f\,(x\,(j-1))-x\,(j-1)\right]$, where, t is an integer, $x(0)$ is an initial condition, $0<\alpha<1$ and $f(x)$ is a difference equation or map. To simplify this notation, we define,$g_{\alpha}\,(t-j)=\frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)}$.

Thus$x\,(t)=x\,(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{t}g_{\alpha}\,(t-j)\left[f\,(x\,(j-1))-x\,(j-1)\right]$, Often-employed definition of one-dimensional coupled map lattice is following. Let $x(i, t)$ be a real variable associated with site i at time t. The evolution is usually given by,

$$\begin{aligned} x(i,t) &= (1-\epsilon) f(x(i,t-1)) \\ &\quad + \frac{\epsilon}{2}[f(x(i+1,t-1)) + f(x(i-1,t-1))], \end{aligned}$$

where, $1 < i < N$ and ϵ is the coupling strength.

We generalize this definition and define one-dimensional coupled fractional maps in the next section. The uncoupled system evolves like N independent fractional maps, *i.e.* for $\epsilon = 0$, the evolution at each site is described by a single fractional map. We also study synchronization and other spatiotemporal patterns in this one-dimensional locally coupled fractional maps. We extend this definition to globally coupled maps in the third section. Similar results are obtained for small-world networks. For non-local couplings, we observe synchronization over a large range of parameter space. The approach to synchronization is not exponential but a power-law. The power is found to be related to the fractional-order parameter. With local coupling, synchronized periodic states are observed. There are clear differences even in the qualitative behaviour of dynamics in presence of long-term memory.

Section snippets

Coupled fractional gauss maps

The discrete Gauss map $f(x)$ is defined as $f(x) = \exp\left(-\nu x^2\right) + \beta$.

We fix $\nu = -7.5$ and vary β. The discrete fractional Gauss map according to the definition in [35] is given as,

$$\begin{aligned} x(t) &= x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha (t-j) \\ &\quad \times \left[\exp\left(-7.5\left(x(j-1)^2\right)\right) + \beta - x(j-1)\right]. \end{aligned}$$

Few comments are in order. Popular maps such as logistic maps or tent maps are a function of unit interval onto itself and coupled map lattice is defined in a way such that coupling keeps the range of values within the unit interval [0,1]. However, the fractional variant of…

Globally coupled fractional gauss maps

We can easily extend the above definition to the high-dimensional lattice. However, it would be cumbersome to simulate such a system and observe patterns in it. Extension of this definition to the mean-field model may shed light on behaviour in a very high dimensional system. We define globally coupled fractional map as,

$$x(i,t) = x(i,0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t-j)\, G_2\left(x(i,j-1), \sum_{k=1}^{N} f(x(k,j-1))\right),$$

where,

$G_2(a,b) = (1-\epsilon)\, f(a) + \frac{\epsilon}{N} b - a.$

We choose Gauss map as the map at a given site and vary the system parameters. We…

Coupled fractional gauss maps with small-world networks

Of late, dynamics on complex networks has been extensively investigated. Most systems such as food web or internet connections do not have a topology of d-dimensional Euclidean space. Small-world, as well as scale-free networks, have been very popular models of complex networks. In small-world networks, we have a high clustering coefficient as well as small path length. We study coupled fractional maps on small-world networks. We begin with a one-dimensional chain where each element is coupled…

Results and discussion

We have extended the definition of fractional maps to a coupled map lattice on arbitrary topology. The system of coupled Gauss maps is analyzed for three different values of the fractional-order parameter, α and different values of map parameter β. We have presented results for locally coupled, globally coupled and small-world networks. For locally coupled Gauss maps, we observe spatially synchronized states with period-3 or period-6 in time. Our studies indicate that such synchronization is…

CRediT authorship contribution statement

Sumit S. Pakhare: Data curation, Visualization, Writing - original draft. **Varsha Daftardar-Gejji:** Data curation, Writing - original draft. **Dilip S. Badwaik:** Data curation. **Amey Deshpande:** Data curation. **Prashant M. Gade:** Conceptualization, Writing - original draft.…

Declaration of Competing Interest

No conflict of interest exists.…

Acknowledgments

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